

# Well-Posedness and Comparison Principle for Option Pricing with Switching Liquidity

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## Abstract

We consider an integro-differential equation derived from a system of coupled parabolic PDE and an ODE which describes an European option pricing with liquidity shocks. We study the well-posedness and prove comparison principle for the corresponding initial value problem.

## 1 Introduction

This work is devoted to the study of an initial value problem of the following form

$$\begin{cases} \frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} &= -\nu_{01}e^{u(S,\tau)} \left( \nu_{10} \int_0^\tau e^{-u(S,s)} ds + e^{-\gamma h(S)} \right) + \kappa, \\ u(S, 0) &= \gamma h(S). \end{cases} \quad (1)$$

Here  $\tau \in [0, T]$ ,  $S \in (0, +\infty)$ ,  $h(S)$  is a given function and  $\sigma$ ,  $\nu_{01}$ ,  $\nu_{10}$ ,  $\kappa$  and  $\gamma$  are constants.

The integro-differential equation in (1) is derived from a system of coupled parabolic PDE and ODE which is suggested by M. Ludkovski and Q. Shen [6] in European option pricing in a financial market switching between two states -a liquid state (0) and an illiquid (1) one. We briefly describe their model. First, it is assumed that the dynamics of the liquidity is represented by a continuous-time Markov chain  $(M_t)$  with intensity rates of the transitions  $0 \rightarrow 1$  and  $1 \rightarrow 0$  and determined by the constants  $\nu_{01}$  and  $\nu_{10}$ , respectively. During the liquid phase ( $M_t = 0$ ) the market dynamics follows the classical Black-Scholes model. More precisely, the price  $S_t$  of a stock is modelled by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with drift  $\mu$  and volatility  $\sigma$  and a standard one-dimensional Brownian motion  $(W_t)$  which is independent of the Markov chain  $(M_t)$  (under the "real world" probability  $\mathbb{P}$ ). Then the wealth process  $(X_t)$  satisfies

$$dX_t = \mu \pi_t X_t dt + \sigma \pi_t X_t dW_t,$$

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where  $\pi_t$  denotes the proportion of stock holdings in the total wealth  $X_t$ . For simplicity, it is assumed that the interest rate of the riskless asset is zero.

Respectively, in the illiquid phase ( $M_t = 1$ ), the market is static and trading in stock is not permitted, i.e.,  $dS_t = dX_t = 0$ .

The presence of liquidity shocks is a source of non-traded risk and makes the market incomplete. Ludkovski and Shen investigate expected utility maximization with exponential utility function:

$$u(x) = -e^{-\gamma x},$$

where  $\gamma > 0$  is the investor's risk aversion parameter. The value functions  $\hat{U}^i(t, X, S)$ ,  $i = 0, 1$  for the optimal investment problem are defined as follows:

$$\hat{U}^i(t, X, S) := \sup_{\pi_t} \mathbb{E}_{t, X, S, i}^{\mathbb{P}} \left[ -e^{-\gamma(X_T + h(S_T))} \right], \quad i = 0, 1,$$

where  $\mathbb{E}_{t, X, S, i}^{\mathbb{P}}$  is the expectation under the measure  $\mathbb{P}$  with starting values  $S_t = S$ ,  $X_t = X$  and  $M_t = i$ . The supremum above is taken over all admissible trading strategies ( $\pi_t$ ) and the function  $h(S)$  denotes the terminal payoff of a contingent claim. Standard stochastic control methods and the properties of the exponential utility function imply that the value functions can be presented by

$$\hat{U}^i(t, X, S) = -e^{-\gamma X} e^{-\gamma R^i(t, S)}, \quad i = 0, 1,$$

where  $R^i(t, S)$  are the unique viscosity solutions of the system ([6])

$$\begin{cases} R_t^0 + \frac{1}{2}\sigma^2 S^2 R_{SS}^0 - \frac{\nu_{01}}{\gamma} e^{-\gamma(R^1 - R^0)} + \frac{d_0 + \nu_{01}}{\gamma} = 0, \\ R_t^1 - \frac{\nu_{10}}{\gamma} e^{-\gamma(R^0 - R^1)} + \frac{\nu_{10}}{\gamma} = 0, \end{cases} \quad (2)$$

with the terminal condition  $R^i(T, S) = h(S)$ ,  $i = 0, 1$ . Here  $d_0 := \mu^2/2\sigma^2$ .

Let  $p$  and  $q$  denote the buyer's indifference prices corresponding to liquid and illiquid initial state respectively. They are defined as follows:  $\hat{U}^0(t, X - p, S) = \hat{V}^0(t, X)$  and  $\hat{U}^1(t, X - q, S) = \hat{V}^1(t, X)$  where  $\hat{V}^i$ ,  $i = 0, 1$  are the value functions of the Merton optimal investment problem (i.e. the case when  $h(S) \equiv 0$ ). It can be shown that  $p$  and  $q$  satisfy a system of differential equations which is quite similar to (2) (see (15)). In fact,

$$p = R^0 + \gamma^{-1} \ln F_0(t) \quad \text{and} \quad q = R^1 + \gamma^{-1} \ln F_1(t)$$

where

$$\begin{aligned} F_0(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ F_1(t) &= \frac{1}{\nu_{01}} (c_1 (d_0 + \nu_{01} - \lambda_1) e^{\lambda_1 t} + c_2 (d_0 + \nu_{01} - \lambda_2) e^{\lambda_2 t}) \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{d_0 + \nu_{01} + \nu_{10} \pm \sqrt{(d_0 + \nu_{01} + \nu_{10})^2 - 4d_0\nu_{10}}}{2}, \\ c_1 &= \frac{\lambda_2 - d_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 T}, \quad \text{and} \quad c_2 = \frac{\lambda_1 - d_0}{\lambda_1 - \lambda_2} e^{-\lambda_2 T}. \end{aligned}$$

Indifference pricing was first used in the pioneering paper of Hodges and Neuberger [3]. We refer also to [2] for further applications (see [4] and [8] as well).

The existence of classical solutions was proved in [6] when the payoff function  $h(S)$  is bounded. This case is restrictive since it does not include such typical example as the call option  $h = \max\{S - K, 0\}$  with strike price  $K$ . We investigate the solvability of the problem and prove the existence and uniqueness of a weak solution in suitable Sobolev weighted spaces which allows unbounded terminal payoff functions.

The integro-differential equation (1) is derived from (2) as follows. Denote  $r^0 := \gamma R^0$ ,  $r^1 = \gamma R^1$ . The system of differential equations for  $r^0$  and  $r^1$  has the following form:

$$\begin{cases} r_\tau^0 - \frac{1}{2}\sigma^2 S^2 r_{SS}^0 = -\nu_{01}e^{-(r^1-r^0)} + d_0 + \nu_{01} \\ r_\tau^1 = -\nu_{10}e^{-(r^0-r^1)} + \nu_{10} \end{cases} \quad (3)$$

where  $\tau = T - t$ . The ODE in (3) can be solved explicitly with respect to  $r^1$ . Then we obtain the initial value problem (1) under the substitution  $u := r^0 - \nu_{10}\tau$  and  $\kappa := d_0 + \nu_{01} - \nu_{10}$ .

The paper is organized as follows. In Section 2 we prove a comparison principle (Theorem 2.1) for classical solutions to the problem (1). Then, in Section 3 we prove a comparison principle (Theorem 3.4) for weak sub/super solutions. In addition, we study the existence and uniqueness of weak solutions in a suitable weighted Sobolev space (see Theorem 3.7).

## 2 Comparison principle for classical solutions

In this section we consider solutions of (1) satisfying

$$|u|, |h| \leq A \exp(\alpha \ln^2 S) = AS^{\alpha \ln S}, \quad (4)$$

for some positive constants  $A$  and  $\alpha$ . Note that conditions (4) include for example linear growth, polynomial and powers of  $S$  with arbitrary exponent.

We prove the following comparison principle:

**Theorem 2.1.** *Let  $u_1, u_0 \in C((0, +\infty) \times [0, T)) \cap C^{2,1}((0, +\infty) \times (0, T))$  be two classical solutions of (1) corresponding to the initial data  $h = h_1$  and  $h = h_0$ , respectively and such that the conditions (4) hold. Then*

$$\gamma \inf (h_1 - h_0) \leq u_1 - u_0 \leq \gamma \sup (h_1 - h_0). \quad (5)$$

We will only prove the lower bound in (5) since the upper one follows immediately from it. In addition, we can assume that

$$\underline{h} := \gamma \inf (h_1 - h_0) > -\infty,$$

otherwise the left inequality in (5) is trivial. We will use the following auxiliary lemma

**Lemma 2.2.** *Let  $u_1$  and  $u_0$  be as in Theorem 2.1 and  $\tau_1 \geq 0$  be such that  $u_1(S, \tau) - u_0(S, \tau) \geq \underline{h}$  for any  $\tau \in [0, \tau_1]$ . Then, there exists a constant  $\bar{\tau} > 0$  such that  $u_1(S, \tau) - u_0(S, \tau) \geq \underline{h}$  for any  $\tau \in [0, \tau_1 + \bar{\tau}]$ . In addition,  $\bar{\tau}$  depends only on  $\alpha$  defined in (4) and  $\sigma$ .*

*Proof.* Let  $u_1$  and  $u_0$  be two solutions of (1) corresponding to the initial conditions  $u_1(S, 0) = \gamma h_1(S)$  and  $u_0(S, 0) = \gamma h_0(S)$ . Denote  $\tilde{u} = u_1 - u_0$ ,  $\tilde{h} = \gamma(h_1 - h_0)$ ,  $u_\xi = \xi u_1 + (1 - \xi) u_0$ ,  $h_\xi = \xi h_1 + (1 - \xi) h_0$ , for  $\xi \in [0, 1]$  and define

$$\mathcal{F}[\tau; u, g] := -\nu_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} ds + e^{-g} \right) + \kappa.$$

Then

$$\mathcal{F}[\tau; u_1, \gamma h_1] - \mathcal{F}[\tau; u_0, \gamma h_0] = \int_0^1 \frac{d}{d\xi} (\mathcal{F}[\tau; u_\xi, \gamma h_\xi]) d\xi \quad (6)$$

$$= -\nu_{01} \tilde{u} \int_0^1 e^{u_\xi(\tau)} \left( \nu_{10} \int_0^\tau e^{-u_\xi(s)} ds + e^{-\gamma h_\xi} \right) d\xi \quad (7)$$

$$+ \nu_{01} \int_0^1 e^{u_\xi(\tau)} \left( \nu_{10} \int_0^\tau e^{-u_\xi(s)} \tilde{u}(s) ds + e^{-\gamma h_\xi} \tilde{h} \right) d\xi$$

$$= -\nu_{01} \nu_{10} \int_0^1 \int_0^\tau e^{u_\xi(\tau) - u_\xi(s)} (\tilde{u}(\tau) - \tilde{u}(s)) ds d\xi \quad (8)$$

$$- \nu_{01} (\tilde{u}(\tau) - \tilde{h}) \int_0^1 e^{u_\xi(\tau) - \gamma h_\xi} d\xi$$

and

$$\tilde{u}_\tau - \frac{1}{2} \sigma^2 S^2 \tilde{u}_{SS} = -\nu_{01} \nu_{10} \int_0^\tau (\tilde{u}(\tau) - \tilde{u}(s)) ds \int_0^1 e^{u_\xi(\tau) - u_\xi(s)} d\xi \quad (9)$$

$$- \nu_{01} (\tilde{u}(\tau) - \tilde{h}) \int_0^1 e^{u_\xi(\tau) - \gamma h_\xi} d\xi$$

Next, define

$$\omega(S, \tau) := \frac{1}{\sqrt{T_1 - \tau}} \exp \left( \frac{(\ln S - \frac{1}{2} \sigma^2 (T_1 - \tau))^2}{2 \sigma^2 (T_1 - \tau)} \right), \quad (10)$$

where  $T_1 > 0$  and  $(S, \tau) \in (0, +\infty) \times [0, T_1)$ . Note that  $\mathcal{L}_{BS} \omega = \omega_\tau - \frac{1}{2} \sigma^2 S^2 \omega_{SS} = 0$  and  $\omega$  is increasing with respect to  $\tau$  in the interval  $\tau \in [T_1 - 4/\sigma^2, T_1)$ . Choose  $T_1 > \tau_1$  in (10) such that the inequality

$$\alpha < \frac{1}{2 \sigma^2 (T_1 - \tau)},$$

holds for all  $\tau \in [\tau_1, T_1)$  and  $T_1 - 4/\sigma^2 < \tau_1$ . It is enough to define  $T_1 := \tau_1 + \bar{\tau}$ ,

where  $0 < \bar{\tau} < \min \left\{ (2\sigma^2\alpha)^{-1}, 4/\sigma^2 \right\}$ . Next, let  $\varphi_\epsilon = \tilde{u} + \epsilon\omega$ . Then

$$\begin{aligned} (\varphi_\epsilon)_\tau - \frac{1}{2}\sigma^2 S^2 (\varphi_\epsilon)_{SS} &= -\nu_{01}\nu_{10} \int_0^\tau (\tilde{u}(\tau) - \tilde{u}(s)) ds \int_0^1 e^{u_\xi(\tau) - u_\xi(s)} d\xi \\ &\quad - \nu_{01} \left( \tilde{u}(\tau) - \tilde{h} \right) \int_0^1 e^{u_\xi(\tau) - \gamma h_\xi} d\xi \end{aligned} \quad (11)$$

$$\begin{aligned} &\geq -\nu_{01}\nu_{10} (\tilde{u}(\tau) - \underline{h}) \int_0^{\tau_1} ds \int_0^1 e^{u_\xi(\tau) - u_\xi(s)} d\xi \\ &\quad - \nu_{01}\nu_{10} \int_{\tau_1}^\tau (\tilde{u}(\tau) - \tilde{u}(s)) ds \int_0^1 e^{u_\xi(\tau) - u_\xi(s)} d\xi \\ &\quad - \nu_{01} \left( \tilde{u}(\tau) - \tilde{h} \right) \int_0^1 e^{u_\xi(\tau) - \gamma h_\xi} d\xi \end{aligned} \quad (12)$$

We will prove that  $\varphi_\epsilon \geq \underline{h}$  for any  $\tau \in [\tau_1, T_1]$ . Indeed, assume by contradiction that  $\inf \varphi_\epsilon < \underline{h}$ . Note that  $\varphi_\epsilon|_{\tau=\tau_1} > \underline{h}$  and there exist  $\bar{S}$  and  $\underline{S}$  such that  $\varphi_\epsilon > \underline{h}$  if either  $S \leq \underline{S}$  or  $S \geq \bar{S}$ . In fact,  $\varphi_\epsilon \rightarrow +\infty$  uniformly when either  $|\ln S| \rightarrow +\infty$  or  $\tau \rightarrow T_1$ . The last observations imply that  $\varphi_\epsilon$  attains minimum in an interior point  $(S_*, \tau_*) \in (\underline{S}, \bar{S}) \times (\tau_1, T_1)$  and  $\varphi_\epsilon(S_*, \tau_*) < \underline{h}$ . Then,  $(\varphi_\epsilon)_\tau(S_*, \tau_*) = 0$ ,  $(\varphi_\epsilon)_{SS}(S_*, \tau_*) \geq 0$  and

$$\tilde{u}(S_*, \tau_*) - \tilde{h} \leq \tilde{u}(S_*, \tau_*) - \underline{h} = \varphi_\epsilon(S_*, \tau_*) - \underline{h} - \epsilon\omega(S_*, \tau_*) < 0 \quad (13)$$

$$\begin{aligned} \tilde{u}(S_*, \tau_*) - \tilde{u}(S_*, s) &= \varphi_\epsilon(S_*, \tau_*) - \varphi_\epsilon(S_*, s) \\ &\quad - \epsilon(\omega(S_*, \tau_*) - \omega(S_*, s)) < 0, \quad \forall s \in [\tau_1, \tau_*], \end{aligned} \quad (14)$$

since  $\omega$  is increasing in  $\tau$ . Thus the right hand side of (12) is positive, a contradiction. Hence  $\varphi_\epsilon = \tilde{u} + \epsilon\omega \geq \underline{h}$  for any  $\tau \in [\tau_1, T_1]$ . Let  $\epsilon \rightarrow 0$ . Then  $\tilde{u} = u_1 - u_0 \geq \underline{h}$  for any  $\tau \in [\tau_1, T_1]$ .  $\square$

*Proof.* (of Theorem 2.1) The comparison principle follows by induction and the auxiliary Lemma 2.2: we first take  $\tau_1 = 0$  and prove it in the interval  $[0, 1/2\bar{\tau}]$ , then let  $\tau_1 = 1/2\bar{\tau}$  and consider the interval  $[1/2\bar{\tau}, \bar{\tau}]$  and etc.  $\square$

Now, as a corollary we formulate comparison principle for the buyer's indifference prices  $p(S, t)$ ,  $q(S, t)$  which satisfy the terminal value problem

$$\begin{cases} p_t + \frac{1}{2}\sigma^2 S^2 p_{SS} - \frac{\nu_{01}}{\gamma} \frac{F_1}{F_0} e^{-\gamma(q-p)} + \frac{d_0 + \nu_{01}}{\gamma} - \frac{1}{\gamma} \frac{F'_0}{F_0} = 0 \\ q_t - \frac{\nu_{10}}{\gamma} \frac{F_0}{F_1} e^{-\gamma(p-q)} + \frac{\nu_{10}}{\gamma} - \frac{1}{\gamma} \frac{F'_1}{F_1} = 0 \\ p(S, T) = q(S, T) = h(S). \end{cases} \quad (15)$$

By *classical solutions* of (15) we mean functions such that  $p \in C((0, +\infty) \times (0, T]) \cap C^{2,1}((0, +\infty) \times (0, T))$ ,  $q \in C((0, +\infty) \times (0, T])$ ,  $q_t \in C((0, +\infty) \times (0, T))$ .

Note that

$$\gamma p = \nu_{10}(T - t) + \ln F_0(t) + u(S, T - t), \quad (16)$$

$$\gamma q = \nu_{10}(T - t) + \ln F_1(t) - \ln \left( \nu_{10} \int_0^{T-t} e^{-u(S, s)} ds + e^{-\gamma h(S)} \right), \quad (17)$$

since  $p(t) = \gamma^{-1} (r^0 + \ln F_0(t))$  and  $q(t) = \gamma^{-1} (r^1 + \ln F_1(t))$ . Then, a comparison principle in  $(p, q)$  solutions will be equivalent to a comparison principle for the  $(r^0, r^1)$  variables.

We consider growth conditions analogous to (4)

$$|p|, |h| \leq A \exp(\alpha \ln^2 S) = AS^{\alpha \ln S}, \quad (18)$$

for some positive constants  $A$  and  $\alpha$ .

**Corollary 2.3.** *Let  $(p_1, q_1)$  and  $(p_0, q_0)$  be two classical solutions of the system (15) corresponding to terminal data  $h \equiv h_1(S)$  and  $h \equiv h_0(S)$ , respectively. If there exist some positive constants  $A$  and  $\alpha$  such that  $p_i(S, t)$  and  $h_i(S)$ ,  $i = 0, 1$  satisfy the conditions (18), then*

$$\inf(h_1 - h_0) \leq p_1(S, t) - p_0(S, t) \leq \sup(h_1 - h_0), \quad (19)$$

$$\inf(h_1 - h_0) \leq q_1(S, t) - q_0(S, t) \leq \sup(h_1 - h_0). \quad (20)$$

*In particular, let  $h(S)$  be bounded from below (or from above) by a constant, i.e.  $h(S) \geq h_*$  (resp.  $h(S) \leq h^*$ ) and  $p(S, t)$ ,  $q(S, t)$ , be a classical solutions of the terminal value problem (15) satisfying (18). Then*

$$p(S, t) \geq h_* \text{ and } q(S, t) \geq h_* \text{ (respectively } p(S, t) \leq h^* \text{ and } q(S, t) \leq h^*),$$

*for any  $S \in (0, +\infty)$  and any  $t \in (0, T]$ .*

*Proof.* The inequalities (19) follow immediately from Theorem 2.1 and representation (16). In order to prove (20) we will use (17), i.e.

$$q_i(\cdot, t) = \gamma^{-1} \left[ \nu_{10}(T - t) + \ln F_1(t) - \ln \left( \nu_{10} \int_0^{T-t} e^{-u_i(\cdot, s)} ds + e^{-\gamma h_i(\cdot)} \right) \right],$$

for  $i = 0, 1$ . Similarly to the proof of Lemma 2.2 we derive

$$\begin{aligned} q_1(\cdot, t) - q_0(\cdot, t) &= -\gamma^{-1} \int_0^1 \frac{d}{d\xi} \left[ \ln \left( \nu_{10} \int_0^{T-t} e^{-u_\xi(\cdot, s)} ds + e^{-\gamma h_\xi(\cdot)} \right) \right] d\xi \\ &= \gamma^{-1} \int_0^1 \frac{\nu_{10} \int_0^{T-t} e^{-u_\xi(\cdot, s)} (u_1(\cdot, s) - u_0(\cdot, s)) ds}{\nu_{10} \int_0^{T-t} e^{-u_\xi(\cdot, s)} ds + e^{-\gamma h_\xi(\cdot)}} d\xi \\ &\quad + (h_1(\cdot) - h_0(\cdot)) \int_0^1 \frac{e^{-\gamma h_\xi(\cdot)}}{\nu_{10} \int_0^{T-t} e^{-u_\xi(\cdot, s)} ds + e^{-\gamma h_\xi(\cdot)}} d\xi \end{aligned}$$

Now, (5) implies the estimates (20).

The second part follows immediately due to the fact that  $p_*(S, t) \equiv h_*$  and  $q_*(S, t) \equiv h_*$  are the solutions of the problem (15) with constant terminal condition  $h \equiv h_*$ . Indeed, if we formally substitute  $p_*(S, t) \equiv h_*$  and  $q_*(S, t) \equiv h_*$  in (15), then we arrive at the conclusion that it is sufficient to check the following identities

$$-\frac{\nu_{01}}{\gamma} \frac{F_1}{F_0} + \frac{d_0 + \nu_{01}}{\gamma} - \frac{1}{\gamma} \frac{F_0'}{F_0} = 0, \quad (21)$$

$$-\frac{\nu_{10}}{\gamma} \frac{F_0}{F_1} + \frac{\nu_{10}}{\gamma} - \frac{1}{\gamma} \frac{F_1'}{F_1} = 0, \quad (22)$$

or equivalently

$$F'_0 = -\nu_{01}F_1 + (d_0 + \nu_{01})F_0, \quad (23)$$

$$F'_1 = -\nu_{10}F_0 + \nu_{10}F_1, \quad (24)$$

which follow directly from the definition of  $F_0$  and  $F_1$ .  $\square$

### 3 Existence of weak solutions

In this section we study the existence and uniqueness of weak solutions in suitable function spaces. First we introduce the weighted  $L^2$  space

$$L_w^2 := \left\{ u : \|u\|_0^2 := \int_0^{+\infty} u^2(S)w(S)dS < \infty \right\},$$

given a weight function  $w > 0$ . Then we define a weighted Sobolev space as follows

$$H_w^1 := \{ u : u \in L_w^2 \text{ s.t. } Su'(S) \in L_w^2 \},$$

with norm  $\|\cdot\|_1$  such that  $\|u\|_1^2 = \|u\|_0^2 + \|Su'\|_0^2$ .

Let  $\xi : [0, +\infty) \rightarrow [0, 1]$  be increasing, infinitely continuously differentiable function and such that  $\xi \equiv 0$  on  $[0, 1/2]$  and  $\xi \equiv 1$  on  $[1, +\infty)$ . We will use  $\xi$  to construct a sequence  $\{u_\epsilon\}$  of compactly supported functions converging in  $H_w^1$  to a given element  $u \in H_w^1$ . More precisely, the following auxiliary result holds.

**Lemma 3.1.** *Let  $\xi_\epsilon(x) := \xi(x/\epsilon)[1 - \xi(x\epsilon/2)]$ ,  $0 < \epsilon < 1$  and  $u_\epsilon := \xi_\epsilon u$ . Then  $u_\epsilon \rightarrow u$  in  $H_w^1$ , as  $\epsilon \rightarrow 0$ .*

*Proof.* Note that  $(u - u_\epsilon)' = (1 - \xi_\epsilon)u' - \xi'_\epsilon u$ ,

$$S\xi'_\epsilon(S) = (S/\epsilon)\xi'(S/\epsilon)[1 - \xi(S\epsilon/2)] - (S\epsilon/2)\xi'(S\epsilon/2)\xi(S/\epsilon)$$

is uniformly bounded with respect to  $\epsilon$  and  $1 - \xi_\epsilon \rightarrow 0$  as well as  $S\xi'_\epsilon(S) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Then the Lebesgue's dominated convergence theorem implies that  $\|u - u_\epsilon\| \rightarrow 0$  as  $\epsilon \rightarrow 0$ .  $\square$

Next, let  $u(S)$  be twice continuously differentiable on  $(0, +\infty)$  and denote the operator  $\mathcal{L}u := -\frac{1}{2}\sigma^2 S^2 u''$ . Then after integration by parts we formally obtain:

$$\begin{aligned} (\mathcal{L}u, v)_{L_w^2} &= -\frac{1}{2}\sigma^2 \int_0^{+\infty} w S^2 u'' v dS \\ &= \frac{1}{2}\sigma^2 \int_0^{+\infty} \left[ w S^2 u' v' + \left( S \frac{w'}{w} + 2 \right) w S u' v \right] dS, \end{aligned}$$

provided that the integrals above are well-defined,  $w$  is continuously differentiable and  $w S^2 u' v \rightarrow 0$  as  $S \rightarrow 0$  and  $S \rightarrow \infty$ . For example, the above holds when  $v$  is continuously differentiable and with compact support.

Following the above observations we introduce the bilinear form:

$$a(u, v) := \frac{1}{2}\sigma^2 \int_0^{+\infty} w S u' \left[ S v' + \left( S \frac{w'}{w} + 2 \right) v \right] dS. \quad (25)$$

If the weight function  $w$  is twice continuously differentiable, and there exists a constant  $C > 0$ , such that

$$\left| S \frac{w'(S)}{w(S)} \right|, \left| S^2 \frac{w''(S)}{w(S)} \right| \leq C, \forall S \in (0, +\infty). \quad (26)$$

then the bilinear form  $a(u, v)$  is continuous and semi-coercive on  $H_w^1$ , i.e.,

$$|a(u, v)| \leq c \|u\|_1 \|v\|_1, \quad \forall u, v \in H_w^1 \quad (27)$$

$$a(u, u) \geq \alpha \|u\|_1^2 - \beta \|u\|_0^2, \quad \forall u \in H_w^1 \quad (28)$$

for some suitable constants  $c > 0$ ,  $\alpha > 0$  and  $\beta > 0$  which are independent of  $u$  and  $v$ .

We can choose such weight function that the call option payoff function  $h = \max\{S - K, 0\}$  belongs to the space  $H_w^1$ , for example, take  $w := (1 + S)^\gamma$ , where  $\gamma < -3$ .

In addition, we assume that

$$\theta := \int_0^{+\infty} w(S) dS < +\infty. \quad (29)$$

This assumption guarantees that any bounded and measurable function belongs to  $L_w^2$ .

**Lemma 3.2.** *There exists a constant  $c_0 > 0$  such that*

$$|u(S)|^2 \leq c_0 \|u\|_1^2 \frac{1}{S} \exp(C |\ln S|), \quad \forall u \in H_w^1, \quad (30)$$

where  $C$  satisfies (26).

*Proof.* Note that there exists a constant  $c_0$  such that

$$|u(1)|^2 \leq c_0 \|u\|_1^2, \quad \forall u \in H_w^1, \quad (31)$$

due to the Sobolev embedding theorem.

Let  $S$  be fixed and denote  $v(\zeta) := u(\zeta S)$ . We have

$$\|v\|_1^2 = \int_0^{+\infty} w(\zeta) \left( \zeta^2 S^2 (u'(\zeta S))^2 + u^2(\zeta S) \right) d\zeta \quad (32)$$

$$= \int_0^{+\infty} \frac{w(\zeta)}{S w(\zeta S)} w(\zeta S) \left( \zeta^2 S^2 (u'(\zeta S))^2 + u^2(\zeta S) \right) d(S\zeta) \quad (33)$$

$$\leq \frac{1}{S} \exp(C |\ln S|) \|u\|_1^2, \quad (34)$$

since

$$\frac{w(\zeta)}{S w(\zeta S)} = \frac{1}{S} \exp \left( \int_{\zeta S}^{\zeta} \frac{w'(\xi)}{w(\xi)} d\xi \right) \leq \frac{1}{S} \exp(C |\ln S|).$$

Then (30) follows from (31) since  $v(1) = u(S)$ .  $\square$



The space  $H_w^1$  is densely and continuously embedded in  $L_w^2$ . We consider the Gelfand triples

$$H_w^1 \subset L_w^2 \subset H_w^*,$$

and

$$L^2(0, T; H_w^1) \subset L^2(0, T; L_w^2) \subset L^2(0, T; H_w^*),$$

where  $H_w^*$  is the dual of  $H_w^1$ . Next, we define the set

$$W(0, T) := \{u \in L^2(0, T; H_w^1), \dot{u} \in L^2(0, T; H_w^*)\}, \quad (35)$$

where  $\dot{u}$  is the distributional derivative of  $u$ . It is well known (see Lions and Magenes[5]) that

$$W(0, T) \subset C([0, T], L_w^2).$$

For simplicity we will further write  $u(\tau)$  instead of  $u(S, \tau)$  when this does not lead to misunderstanding. Recall that

$$\mathcal{F}[\tau; u, \gamma h] := -\nu_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} ds + e^{-\gamma h} \right) + \kappa.$$

**Definition 3.3.** A function  $u \in W(0, T)$  is called weak supersolution (subsolution) of the initial value problem (1) if  $u(0) \geq \gamma h$  (resp.  $u(0) \leq \gamma h$ ) and for a.a.  $\tau \in (0, T)$  the inequality

$$\langle \dot{u}, v \rangle + a(u, v) \geq (\leq) \int_0^{+\infty} w \mathcal{F}[\tau; u, \gamma h] v dS, \quad (36)$$

holds for any nonnegative  $v \in H_w^1$ . Respectively, the function  $u \in W(0, T)$  is called weak solution of the initial value problem (1) if  $u(0) = \gamma h$  and for a.a.  $\tau \in (0, T)$  the equality

$$\langle \dot{u}, v \rangle + a(u, v) = \int_0^{+\infty} w \mathcal{F}[\tau; u, \gamma h] v dS, \quad \forall v \in H_w^1, \quad (37)$$

holds.

Next, we prove the following comparison principle for weak super/subsolutions satisfying growth conditions of type (4).

**Theorem 3.4.** Let  $\bar{u}$  be a weak supersolution of the initial value problem (1) with initial data  $h(S) \equiv \bar{h}$  and  $\underline{u}$  be a weak subsolution corresponding to the initial data  $h(S) \equiv \underline{h}$  where  $\underline{h}$  and  $\bar{h}$  are given and  $\underline{h} \leq \bar{h}$ . Assume in addition, that there exist positive constants  $A$  and  $\alpha$  such that

$$|\underline{h}|, |\bar{h}|, |\underline{u}|, |\bar{u}| \leq A \exp(\alpha \ln^2 S) = AS^{\alpha \ln S}, \quad (38)$$

for a.a.  $(S, t) \in (0, +\infty) \times [0, T]$ .

Then  $\underline{u} \leq \bar{u}$  for a.a.  $(S, t) \in (0, +\infty) \times [0, T]$ .

Denote  $u := \bar{u} - \underline{u}$ . We will prove that  $u_- := \max\{-u, 0\} = 0$  almost everywhere. Similarly to (9), we obtain that the following inequality holds for a.a.  $\tau \in (0, T)$  and for any nonnegative  $v \in H_w^1$  with compact support in  $(0, +\infty)$ :

$$\begin{aligned} \langle \dot{u}, v \rangle + a(u, v) &\geq -\nu_{01} \nu_{10} \int_0^\infty \left( \int_0^\tau \delta(\tau, s) (u(S, \tau) - u(S, s)) ds \right) v(S) w dS \\ &\quad - \nu_{01} \int_0^\infty \left( u(S, \tau) - \tilde{h}(S) \right) v(S) \delta(\tau) w dS, \end{aligned} \quad (39)$$

where

$$\delta(\tau, s) := \int_0^1 e^{u_\xi(\tau) - u_\xi(s)} d\xi, \quad \delta(\tau) := \int_0^1 e^{u_\xi(\tau) - \gamma h_\xi} d\xi,$$

$$u_\xi := \xi \bar{u} + (1 - \xi) \underline{u}, \quad u(\cdot, 0) \geq \tilde{h} := \gamma(\bar{h} - \underline{h}) \geq 0 \text{ and } h_\xi := \xi \bar{h} + (1 - \xi) \underline{h}.$$

It is sufficient to prove the following auxiliary result:

**Lemma 3.5.** *Assume that  $\tau_1 \geq 0$  is such that for any  $t \in [0, \tau_1]$  the inequality  $\bar{u}(t) - \underline{u}(t) \geq 0$  holds a.e. on  $(0, +\infty)$ . Then the same inequality holds for any  $t \in [0, \tau_1 + \bar{\tau}]$ , where  $\bar{\tau} > 0$  is a constant which depends only on  $\alpha$  and  $\sigma$ .*

*Proof.* Let  $\omega$  be defined by (10) and  $u_\epsilon := u + \epsilon \omega$  where  $u = \bar{u} - \underline{u}$ . Then, assume that  $\bar{\tau}$  is chosen as in the proof of Lemma 2.2. We will prove that  $u_{\epsilon-} := \max\{-u_\epsilon, 0\} \equiv 0$  for a.a.  $(S, t) \in (0, +\infty) \times [\tau_1, \tau_1 + \bar{\tau}]$ . Note that there exist a closed interval  $I_\epsilon \subset (0, +\infty)$  such that  $u_{\epsilon-} = 0$  on the set  $((0, +\infty) \setminus I_\epsilon) \times [\tau_1, \tau_1 + \bar{\tau}]$  due to the conditions (38). Now, let  $\varphi(S)$  be a smooth function with compact support in  $(0, +\infty)$  such that  $\varphi(S) = 1$  on the interval  $I_\epsilon$ . Then  $u_\epsilon \varphi \in L^2(\tau_1, \tau_1 + \bar{\tau}; H_w^1)$  and  $(u_\epsilon \varphi)_- = u_{\epsilon-}$ . Next, for any nonnegative  $v \in H_w^1$  with compact support  $\text{supp } v \subset I_\epsilon$  we have  $\varphi v = v$ ,  $a(u\varphi, v) = a(u, v)$  and then

$$\left\langle \frac{d}{d\tau} (u_\epsilon \varphi), v \right\rangle + a(u_\epsilon \varphi, v) = \langle \dot{u}, \varphi v \rangle + \epsilon \langle \varphi \dot{\omega}, v \rangle + a(u\varphi, v) + \epsilon a(\omega\varphi, v) \quad (40)$$

$$= \langle \dot{u}, v \rangle + a(u, v) - \underbrace{\frac{1}{2} \epsilon \sigma^2 (2\omega' \varphi' + \omega \varphi'')}_{=0} v \Big|_{\tau_1}^{\tau_1 + \bar{\tau}} \quad (41)$$

$$\begin{aligned} & \geq -\nu_{01} \nu_{10} \int_0^\infty \left( \int_0^\tau \delta(\tau, s) (u(S, \tau) - u(S, s)) ds \right) v(S) \omega dS \\ & \quad - \nu_{01} \int_0^\infty \left( u(S, \tau) - \tilde{h}(S) \right) v(S) \delta(\tau) \omega dS \\ & \geq -\nu_{01} \nu_{10} \int_0^\infty \left( \int_0^{\tau_1} \delta(\tau, s) ds \right) u(S, \tau) v(S) \omega dS \\ & \quad - \nu_{01} \nu_{10} \int_0^\infty \left( \int_{\tau_1}^\tau \delta(\tau, s) (u(S, \tau) - u(S, s)) ds \right) v(S) \omega dS \\ & \quad - \nu_{01} \int_0^\infty u(S, \tau) v(S) \delta(\tau) \omega dS, \end{aligned} \quad (42)$$

i.e.,

$$\begin{aligned} & \left\langle \frac{d}{d\tau} (u_\epsilon \varphi), v \right\rangle + a(u_\epsilon \varphi, v) \geq -\nu_{01} \nu_{10} \int_0^\infty \left( \int_0^{\tau_1} \delta(\tau, s) ds \right) u_\epsilon(S, \tau) v(S) \omega dS \\ & \quad - \nu_{01} \nu_{10} \int_0^\infty \left( \int_{\tau_1}^\tau \delta(\tau, s) (u_\epsilon(S, \tau) - u_\epsilon(S, s)) ds \right) v(S) \omega dS \\ & \quad - \nu_{01} \int_0^\infty u_\epsilon(S, \tau) v(S) \delta(\tau) \omega dS, \end{aligned} \quad (43)$$

where we have used the fact that  $u_\epsilon > u$  and  $u_\epsilon(S, \tau) - u_\epsilon(S, s) > u(S, \tau) - u(S, s)$  for any  $s \in [\tau_1, \tau]$  since  $\omega(S, \cdot)$  is increasing on that interval. Now, take  $v = u_{\epsilon-}$  and note that  $u_\epsilon = u_{\epsilon+} - u_{\epsilon-}$ ,  $a(u_\epsilon \varphi, u_{\epsilon-}) = -a(u_{\epsilon-}, u_{\epsilon-})$  and

$$u_\epsilon(S, s) u_{\epsilon-}(S, \tau) \geq -u_{\epsilon-}(S, s) u_{\epsilon-}(S, \tau) \geq -\frac{1}{2} (u_{\epsilon-}^2(S, s) + u_{\epsilon-}^2(S, \tau)).$$

After integration with respect to  $\tau$  from  $\tau_1$  to  $t \in [\tau_1, \tau_1 + \bar{\tau}]$  the inequality (43) implies

$$\frac{1}{2} \|u_{\epsilon-}(t)\|_0^2 + a(u_{\epsilon-}, u_{\epsilon-}) \leq - \int_{\tau_1}^t \left( \int_0^\infty \Sigma(S, \tau) u_{\epsilon-}^2(S, \tau) w dS \right) d\tau, \quad (44)$$

where

$$\Sigma(S, \tau) := \nu_{01} \nu_{10} \left( \int_0^{\tau_1} \delta(\tau, s) ds + \frac{1}{2} \int_{\tau_1}^\tau \delta(\tau, s) ds - \frac{1}{2} \int_\tau^t \delta(s, \tau) ds \right) + \nu_{01} \delta(\tau).$$

$|\Sigma(S, \tau)|$  is bounded from above by a constant, say  $C > 0$ , when  $S \in I_\epsilon$  and due to the semi-coercivity of the bilinear form  $a(\cdot, \cdot)$  (see (28)) we obtain:

$$\frac{1}{2} \|u_{\epsilon-}(t)\|_0^2 \leq (C + \beta) \int_{\tau_1}^t \|u_{\epsilon-}(\tau)\|_0^2 d\tau. \quad (45)$$

Hence the Gronwall inequality implies  $\|u_{\epsilon-}(t)\|_0 = 0$  for any  $t \in [\tau_1, \tau_1 + \bar{\tau}]$  since  $\|u_{\epsilon-}(\tau_1)\|_0 = 0$ . Then  $u + \epsilon\omega \geq 0$  a.e. Thus  $u \geq 0$  a.e. since  $\epsilon > 0$  is arbitrary.  $\square$

We further prove another useful estimate.

**Lemma 3.6.** *There exists a constant  $C > 0$  such that*

$$\max_{t \in [0, T]} \|u(t)\|_0 + \|u\|_{L^2(0, T, H_w^1)} \leq C \left( \|u(0)\|_0 + \|\hat{u}\|_{W(0, T)} + \gamma \|h\|_0 + 1 \right) \quad (46)$$

for any weak subsolution  $u$  and any function  $\hat{u} \in W(0, T)$  satisfying  $u \geq \hat{u}$ .

*Proof.* Let  $v \in H_w^1$  be some nonnegative function. We have

$$\begin{aligned} \langle \dot{u}, v \rangle + a(u, v) &\leq \int_0^{+\infty} w \mathcal{F}[\tau; u, \gamma h] v dS, \\ &\leq -\nu_{01} \nu_{10} \left( \int_0^\tau [u(\tau) - u(s)] ds, v \right)_{L_w^2} \\ &\quad - \nu_{01} (u(\tau) - \gamma h, v)_{L_w^2} \\ &\quad + (\kappa - \nu_{01} \nu_{10} \tau - \nu_{01}) (1, v)_{L_w^2}. \end{aligned} \quad (47)$$

Take  $v = u - \hat{u}$  and integrate (47) with respect to  $\tau$  from 0 to  $t$ .

$$\begin{aligned} \frac{1}{2} \|u(t)\|_0^2 + a(u, u) &\leq \frac{1}{2} \|u(0)\|_0^2 + (u, \hat{u})_{L_w^2} \Big|_0^t + a(u, \hat{u}) - \int_0^t \langle \dot{u}, u \rangle d\tau \\ &\quad - \nu_{01} \int_0^t (\nu_{10} \tau + 1) \|u(\tau)\|_0^2 d\tau + \nu_{01} \nu_{10} \frac{1}{2} \left\| \int_0^t u(\tau) d\tau \right\|_0^2 \\ &\quad + C_1 \left( \|\hat{u}\|_{L^2(0, t, L_w^2)} + \gamma \|h\|_0 + 1 \right) \|u\|_{L^2(0, t, L_w^2)} \\ &\quad + C_2 (\gamma \|h\|_0 + 1) \|\hat{u}\|_{L^2(0, t, L_w^2)}. \end{aligned} \quad (48)$$

Then a standard argument implies the estimate (46).  $\square$

Now, we prove the existence of weak solutions, provided that  $h \in H_w^1$ . The proof is based on the lower and upper solution method (cf. [7]). However, the exponential nonlinearity in (1) causes some very technical difficulties which have to be overcome.

**Theorem 3.7.** *Assume that  $h \in H_w^1$ . Then there exist a weak solution  $u$  to the initial value problem (1). Moreover, there exists a constant  $C > 0$  independent of  $u$  such that*

$$\|\dot{u}\|_{L^2(0,T,L_w^2)} + \|u\|_{L^\infty(0,T,H_w^1)} \leq C(\|u(0)\|_1 + 1) \quad (49)$$

*Proof.* We will present the proof in several steps.

**Step 1.** *Let  $h \in L_w^2$  be bounded. Then there exists a weak solution  $u$  to the initial value problem (1). In addition, if  $u(0) = \gamma h \in H_w^1$ , then the inequality (49) holds with a constant  $C$  independent of  $u(0)$ .*

Note that we can construct appropriate couple of a supersolution  $\bar{u}$  and a subsolution  $\underline{u}$ . Indeed, let the constant  $c_0$  be such that  $|\gamma h| \leq c_0$  and take  $\underline{u} := -c_0 - Mt$  for some positive constant  $M$ . If  $M$  is great enough then  $\underline{u}$  is a subsolution. Analogously,  $\bar{u} := c_0 + Mt$  is a supersolution provided that  $M \geq \kappa$ . Next, according to (8) we can choose a constant  $N > 0$  such that

$$Nu(\tau) + \mathcal{F}[\tau; u, \gamma h] = Nu(\tau) - \nu_{01}e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} ds + e^{-\gamma h} \right) + \kappa$$

is increasing in  $u$ , i.e.

$$Nu_1(\tau) + \mathcal{F}[\tau; u_1, \gamma h] \geq Nu_0(\tau) + \mathcal{F}[\tau; u_0, \gamma h],$$

for all  $u_0$  and  $u_1$  such that  $\underline{u} \leq u_0 \leq u_1 \leq \bar{u}$ . Now, we can construct a decreasing sequence of supersolutions  $u_0 := \bar{u}, u_1, u_2, \dots$  such that  $u_{n+1}$  is the solution of the initial value problem

$$\begin{cases} \dot{u}_{n+1} - \frac{1}{2}\sigma^2 S^2 u_{n+1,SS}'' + Nu_{n+1} = Nu_n + \mathcal{F}[\tau; u_n, \gamma h], \\ u_{n+1}(S, 0) = \gamma h(S) \end{cases}$$

and  $\underline{u} \leq u_n \leq \bar{u}$ . A standard argument implies that  $u_n$  converges to a weak solution of the problem (1). We omit the details.

Next, assume in addition that  $h \in H_w^1$ . Then  $\dot{u} \in L^2(0, T; L_w^2)$  and  $u \in L^\infty(0, T; H_w^1)$  (see, e.g., Bonnans [1]) and the following parabolic estimate holds:

$$\|\dot{u}\|_{L^2(0,T,L_w^2)} + \|u\|_{L^\infty(0,T,H_w^1)} \leq c_0 \left( \|u(0)\|_1 + \|\mathcal{F}[\cdot; u, \gamma h]\|_{L^2(0,T,L_w^2)} \right)$$

We will prove the stronger estimate (49). First, we have

$$-\frac{1}{2}\sigma^2 S^2 u_{SS}'' = \mathcal{F}[\tau; u, \gamma h] - \dot{u} \in L^2(0, T, L_w^2), \quad (50)$$

$$\begin{aligned} -\frac{1}{2}\sigma^2 \int_0^t (S^2 u_{SS}'', \dot{u})_{L_w^2} d\tau &= \frac{1}{2}\sigma^2 \left( \frac{1}{2} \|u(t)\|_1^2 - \frac{1}{2} \|u(0)\|_1^2 \right) \\ &\quad + \frac{1}{2}\sigma^2 \int_0^t \left( S \left( S \frac{w'}{w} + 2 \right) u_S' - u, \dot{u} \right)_{L_w^2} d\tau \end{aligned} \quad (51)$$

$$\int_0^t (\mathcal{F}[\tau; u, \gamma h], \dot{u})_{L_w^2} d\tau = \int_0^{+\infty} \left( \int_0^t \frac{d}{d\tau} (\mathcal{F}[\tau; u, \gamma h]) d\tau \right) w dS \quad (52)$$

$$\begin{aligned} &+ \int_0^{+\infty} \left( \int_0^t (\kappa \dot{u} + \nu_{01} \nu_{10}) d\tau \right) w dS \\ &\leq |\kappa| \theta^{1/2} \int_0^t \|\dot{u}(\tau)\|_0 d\tau + \nu_{01} (1 + \nu_{10} t) \theta \end{aligned} \quad (53)$$

since

$$\begin{aligned} \frac{d}{d\tau} (\mathcal{F}[\tau; u, \gamma h]) &= \frac{d}{d\tau} \left[ -\nu_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} ds + e^{-\gamma h} \right) + \kappa \right] \\ &= -\nu_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} ds + e^{-\gamma h} \right) \dot{u} - \nu_{01} \nu_{10} \end{aligned} \quad (54)$$

$$= \mathcal{F}[\tau; u, \gamma h] \dot{u} - \kappa \dot{u} - \nu_{01} \nu_{10}. \quad (55)$$

and

$$\int_0^t \frac{d}{d\tau} (\mathcal{F}[\tau; u, \gamma h]) d\tau = \mathcal{F}[t; u, \gamma h] - \mathcal{F}[0; u, \gamma h] \leq \nu_{01} \quad (56)$$

We multiply both sides of the equation  $\dot{u} - 1/2\sigma^2 S^2 u''_{SS} = \mathcal{F}[\tau; u, \gamma h]$  with  $\dot{u}$  in  $L_w^2$  and integrate from 0 to  $T$ . Then (51) and (53) imply

$$\begin{aligned} \int_0^t \|\dot{u}\|_0^2 d\tau + \frac{1}{4}\sigma^2 \|u(t)\|_1^2 &\leq -\frac{1}{2}\sigma^2 \int_0^t \left( S \left( S \frac{w'}{w} + 2 \right) u'_S - u, \dot{u} \right)_{L_w^2} d\tau \quad (57) \\ &+ |\kappa| \theta^{1/2} \int_0^t \|\dot{u}(\tau)\|_0 d\tau + \frac{1}{4}\sigma^2 \|u(0)\|_1^2 \\ &+ \nu_{01} (1 + \nu_{10} t) \theta \\ &\leq \tilde{C} \left[ \int_0^t (\|u(\tau)\|_1 + 1) \|\dot{u}(\tau)\|_0 d\tau + \|u(0)\|_1^2 + 1 \right] \end{aligned}$$

for some constant  $\tilde{C} > 0$ . Now, a technical, but standard argument implies that (49) holds.

**Step 2.** Let  $h \in H_w^1$  be bounded from below, i.e.,  $u(0) = \gamma h \geq c$ . Then there exists a weak solution  $u$  to the initial value problem (1). In addition, the inequality (49) holds.

Let  $\xi_\epsilon(x)$  be defined as in Lemma 3.1, i.e.,  $\xi_\epsilon(x) := \xi(x/\epsilon) [1 - \xi(x\epsilon/2)]$ . Step 1 implies that there exists a solution  $u_\epsilon$  corresponding to the initial condition  $u_\epsilon(0) = \xi_\epsilon(\gamma h - c) + c = \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c$  which is bounded. Moreover,  $\xi_\epsilon \gamma h + (1 - \xi_\epsilon)c \leq \gamma h$  increases as  $\epsilon \downarrow 0$  and converges in  $H_w^1$  to  $\gamma h$ . Then the comparison principle from Theorem 3.4 implies that the sequence  $u_\epsilon$  is increasing as  $\epsilon \downarrow 0$ . Next, the estimate (49) and Lemma 3.2 imply that  $u_\epsilon(S, \tau)$  converges to a finite limit  $u(S, \tau)$  for any  $(S, \tau) \in (0, +\infty) \times [0, T]$ . What is more,  $\dot{u}_\epsilon$  is weakly convergent to  $\dot{u}(S, \tau)$  in  $L^2(0, T; L_w^2)$ ,  $u_\epsilon$  is weakly-\* convergent to  $u$  in  $L^\infty(0, T, H_w^1)$  and  $u$  satisfies the estimate (49). Then it is sufficient to prove that  $\mathcal{F}[\tau; u_\epsilon, \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c]$  is weakly convergent to  $\mathcal{F}[\tau; u, \gamma h]$  in  $L^2(0, T; H_w^*)$ . First, note that

$$\mathcal{F}[\tau; u_\epsilon, \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c] = \dot{u}_\epsilon - \frac{1}{2}\sigma^2 S^2 u''_{\epsilon, SS}$$

is bounded in  $L^2(0, T; H_w^*)$  and then there exists an element  $\tilde{\mathcal{F}} \in L^2(0, T; H_w^*)$  such that

$$\mathcal{F}[\tau; u_\epsilon, \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c] \xrightarrow{L^2(0, T; H_w^*)} \tilde{\mathcal{F}}.$$

On the other hand,  $\mathcal{F}[\tau; u_\epsilon, \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c]$  is bounded from above by the constant function  $\kappa$ . Let  $v \in L^2(0, T; H_w^1)$  be some arbitrary nonnegative function. Then Fatou's lemma implies

$$\begin{aligned} \langle \kappa - \tilde{\mathcal{F}}, v \rangle &= \lim_{\epsilon \rightarrow 0} \langle \kappa - \mathcal{F}[\cdot; u_\epsilon, \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c], v \rangle_{L^2(0, T; L_w^2)} \\ &\geq \langle \kappa - \mathcal{F}[\cdot; u, \gamma h], v \rangle \geq 0, \end{aligned} \quad (58)$$

i.e.

$$\mathcal{F}[\cdot; u, \gamma h] \in L^2(0, T; H_w^*) \text{ and } \mathcal{F}[\cdot; u, \gamma h] \geq \tilde{\mathcal{F}}.$$

Finally, we prove that in fact

$$\mathcal{F}[\cdot; u, \gamma h] \equiv \tilde{\mathcal{F}}, \text{ i.e., } \langle \mathcal{F}[\cdot; u, \gamma h], v \rangle = \langle \tilde{\mathcal{F}}, v \rangle \quad \forall v \in L^2(0, T; H_w^1). \quad (59)$$

First, observe that,  $v_\epsilon := \xi_\epsilon v \rightarrow v$  as  $\epsilon \rightarrow 0$  in  $L^2(0, T; H_w^1)$ . Hence, it is sufficient to prove (59) for functions  $v$  vanishing outside a set of the form  $I \times [0, T]$  where  $I \subset (0, +\infty)$  is some closed interval. According to estimate (49) and Lemma 3.2 (applied to the interval  $I$ ) the functions  $u_\epsilon$  and  $u$  are uniformly bounded on  $I \times [0, T]$ . Then

$$\langle \mathcal{F}[\cdot; u, \gamma h], v \rangle = \lim_{\epsilon \rightarrow 0} \langle \mathcal{F}[\cdot; u_\epsilon, \xi_\epsilon \gamma h + (1 - \xi_\epsilon)c], v \rangle_{L^2(0, T; L_w^2)} = \langle \tilde{\mathcal{F}}, v \rangle.$$

**Step 3.** Let  $h \in H_w^1$ . Then there exists a weak solution  $u$  to the initial value problem (1). In addition, the inequality (49) holds.

Consider a sequence of problems with initial condition

$$u_N(S, 0) = \max \{ \gamma h(S), -N \}, \quad N = 1, 2, \dots$$

Then the corresponding solutions  $u_N$  form a decreasing sequence due to the comparison principle and Lemma 3.2. Moreover, the pointwise limit  $\lim_{N \rightarrow \infty} u_N(S, \tau)$  is finite for any  $(S, \tau)$  since the inequality (49) holds for each function  $u_N$ . Then the proof follows similar arguments as in Step 2.  $\square$

Finally, note that the uniqueness of the weak solution is a consequence of the comparison principle. More precisely, we have the following corollary.

**Corollary 3.8.** Assume that  $h \in H_w^1$ . Then there exists a unique weak solution  $u \in W(0, T) \cap L^\infty(0, T; H_w^1)$  to the initial value problem (1). Moreover, the estimate (49) holds with a constant  $C > 0$  independent of  $u$ .

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